Ex. 7.2. Design a low pass discrete time filter by applying IIM to an appropriate continuous time Butterworth filter. The specs for the discrete-time filter is given by

$$0.89125 \leq |H(e^{j\omega})| \leq 1, \quad 0 \leq |\omega| \leq 0.2\pi$$

$$|H(e^{j\omega})| \leq 0.17783, \quad 0.3\pi \leq |\omega| \leq \pi$$

Sol'n//

$\omega = \Omega T_s$, let $T_s = 1$ (for convenience) $\Rightarrow$ $\omega = \Omega$; the specs become

$$0.89125 \leq |H_a(j\omega)| \leq 1 \quad ; \; 0 \leq |\Omega| \leq 0.2\pi$$

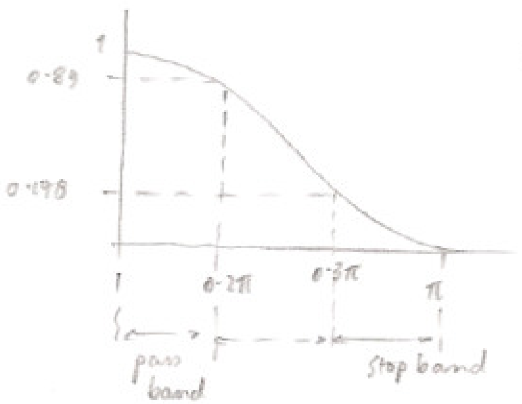
$$|H_a(j\omega)| \leq 0.178 \quad ; \; 0.3\pi \leq |\Omega| \leq \pi$$

We know $|H_a(j\omega)|^2 = \dfrac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$, where Ω_c = cutoff freq.
N = order of filter

Because of monotonic response of analog Butterworth filter:

$$|H_a(j0.2\pi)| = 0.89$$

$$|H_a(j0.3\pi)| = 0.178$$



$\therefore$

$$1 + \left(\frac{0.2\pi}{\Omega_c}\right)^{2N} = \frac{1}{0.89^2} \quad \text{--- (1)} \rightarrow \text{corresponds to passband}$$

$$1 + \left(\frac{0.3\pi}{\Omega_c}\right)^{2N} = \frac{1}{(0.178)^2} \quad \text{--- (2)} \rightarrow \text{corresponds to stop band}$$

Solving (1) & (2), 2 eq. 2 unk.,

$N = 5.8858 \approx 6$ → always round to <u>next highest</u> integer

a 6th order filter

$\Omega_c = 0.7047$

Since N was rounded, (1) & (2) cannot satisfy both equations simultaneously. Ω_c can be chosen to exceed the specified requirements in either the passband or stopband or both. If we substitute $N=6 \rightarrow$ (1), the new $\Omega_c = 0.7032$. With this value, the passband is satisfied and the stopband will be exceeded. This allows some margin for aliasing in the discrete time filter.

$$|H_a(j\omega)|^2 = H_a(s)H_a(-s) = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} = \frac{1}{1+\left(\frac{s}{j\Omega_c}\right)^{2N}} = \left.\frac{(j\Omega_c)^{2N}}{s^{2N}+(j\Omega_c)^{2N}}\right|_{s=j\Omega}$$

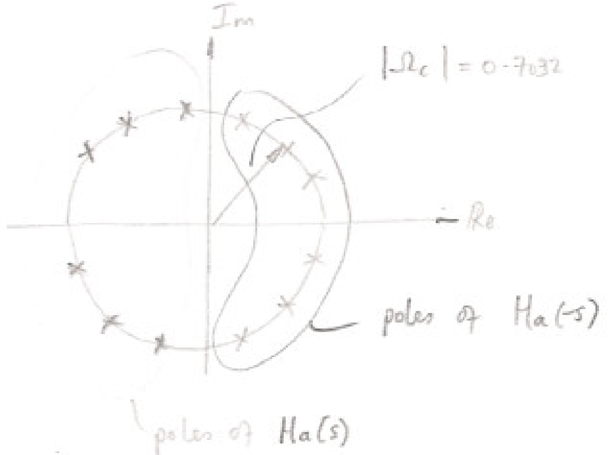
$\left(\Omega = \frac{s}{j}\right)$

The roots of the denominator, ie. the poles, are

$$P_k = (-1)^{1/2N}(j\Omega_c) = \Omega_c e^{j\frac{2\pi}{N}(2k+N+1)} \; ; \quad k = 0, 1, \cdots, 2N-1.$$

Ω_c ↑ root magnitude; ↑ root phase angle

With $N=6$, there are 12 roots distributed evenly in the s-plane (@ 30° intervals)



With $N=6$, $\Omega_c = 0.7032$,

pole pair 1 : $-0.182 \pm j0.679$
2 : $-0.497 \pm j0.497$
3 : $-0.679 \pm j0.182$

$$\therefore \quad H_a(s) = \frac{\overset{\text{from } (0.7032)^6}{0.12093}}{(s+0.182-j0.679)(s+0.182+j0.679)\cdots\cdots(s+0.679-j0.182)(s+0.679+j0.182)}$$

Now we have to express $H_a(s)$ as a partial fraction expansion, and then we can do the transformation.

$$H(z) = \frac{0.2871 - 0.4466z^{-1}}{1 - 1.2971z^{-1} + 0.6949z^{-2}} + \frac{-2.1428 + 1.455z^{-1}}{1 - 1.0691z^{-1} + 0.3699z^{-2}} + \frac{1.8557 + 0.6303z^{-1}}{1 - 0.997z^{-1} + 0.257z^{-2}}$$

Ca use this to find the difference equation.

Ex2 Transform $H_a(s) = \dfrac{s+1}{s^2+5s+6}$ into a digital filter using IIM technique, in which $T_s = 0.1$.

Sol'n

$$H_a(s) = \frac{s+1}{(s+3)(s+2)} \xrightarrow{\text{partial fractions}} \frac{-1}{s+2} + \frac{2}{s+3}$$

$$\left.\begin{array}{l} s_1 = -2 \\ s_2 = -3 \end{array}\right\} \text{ s-domain roots}$$

$$H(z) = \frac{-1}{1-\underbrace{e^{-2\times 0.1}}_{e^{-s_k T_s}} z^{-1}} + \frac{2}{1-e^{-3\times 0.1} z^{-1}} = \frac{1-0.8966 z^{-1}}{1-1.5595 z^{-1} + 0.6065 z^{-2}}$$

$$H(z) = \frac{Y(z)}{X(z)} \Rightarrow \boxed{\underset{a_0}{\uparrow} y[n] - \underset{a_1}{1.5595} y[n-1] + \underset{a_2}{0.6065} y[n-2] = \underset{b_0}{x[n]} - \underset{b_1}{0.8966} x[n-1]}$$

MATLAB FUNCTION:

$$[a, b] = \underset{CF}{BW} \{N, \Omega_c\}$$

Extended question → "Implement the filter using the D-II technique."

Butterworth Polynomial Tables : Table 8.1

Replace s with $\dfrac{s}{\Omega_c}$

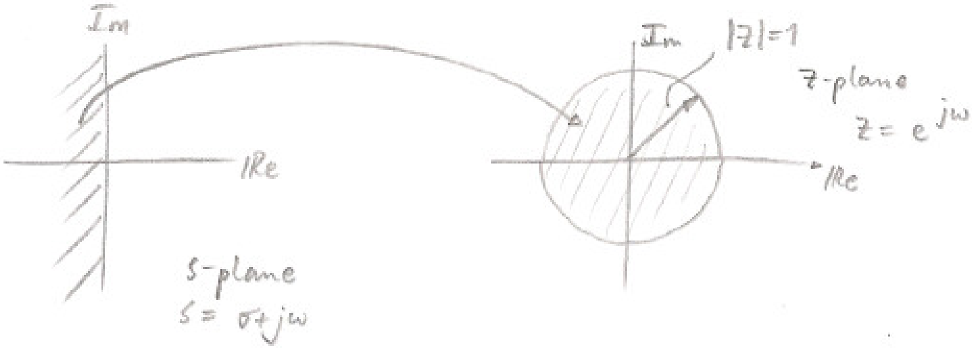
ex for $N=2$,

$$Q(s) = \left(\frac{s}{\Omega_c}\right)^2 + \sqrt{2}\left(\frac{s}{\Omega_c}\right) + 1$$

$$* \quad H(s) = \frac{1}{Q(s)} \quad *$$

8.2. Bilinear Transformation Method

This transformation maps the entire $j\Omega$ axis in the s-plane onto the unit circle in the z-plane.



The relation for BLT is given by:

$$s = \frac{2}{T_s}\,\frac{1-z^{-1}}{1+z^{-1}} \Rightarrow z = \frac{1+{sT_s}/{2}}{1-{sT}/{2}}$$

$$\therefore H(z) = H_a(s)\Big|_{s=\frac{2}{T_s}\cdot\frac{1-z^{-1}}{1+z^{-1}}}$$

this is the most popular transformation.

$$s z \frac{T_s}{2} + s\frac{T_s}{2} - z + 1 = 0$$

this eqn is linear in either s or z if the other variable is held constant $\Rightarrow$ hence "bilinear".

Properties of BLT:

$$z = \frac{1 + \frac{\sigma T_s}{2} + j\frac{\Omega T_s}{2}}{1 - \frac{\sigma T_s}{2} - j\frac{\Omega T_s}{2}}, \quad s = \sigma + j\Omega$$

$$\therefore |z| = \frac{\sqrt{\left(1+\frac{\sigma T_s}{2}\right)^2 + \left(\frac{\Omega T_s}{2}\right)^2}}{\sqrt{\left(1-\frac{\sigma T_s}{2}\right)^2 + \left(\frac{\Omega T_s}{2}\right)^2}}$$

$\therefore$ the $j\Omega$ axis will map onto the unit circle in z-plane in one-to-one fashion thus it avoids aliasing. Only one point from $j\Omega$ axis can map onto each point on the unit circle.

left half in s-plane

(i) If $\sigma < 0$, $|z| < 1$ $\rightarrow$ the entire left half of s-plane maps within the unit circle $\Rightarrow$ Stable transformation.

(ii) If $\sigma > 0$, $|z| > 1$ $\rightarrow$ opposite to case (i) $\Rightarrow$ unstable (outside of unit circle)

(iii) If $\sigma = 0$ $|z| = 1$ $\rightarrow$ marginally stable (on unit stable)

Relation between ω and Ω

$$\omega \neq \Omega T_s$$

$$e^{j\theta k} = (\cos\theta + j\sin\theta)^k$$

$$s = \frac{2}{T_s} \cdot \frac{1-z^{-1}}{1+z^{-1}} = \frac{2}{T_s} \cdot \frac{1-e^{-j\omega}}{1+e^{-j\omega}}$$

$$\Rightarrow s\frac{T_s}{2} = \frac{e^{j\omega/2} - e^{-j\omega/2}}{e^{j\omega/2} + e^{-j\omega/2}}$$

$$= \frac{2j\sin\frac{\omega}{2}}{2j\cos\frac{\omega}{2}}$$

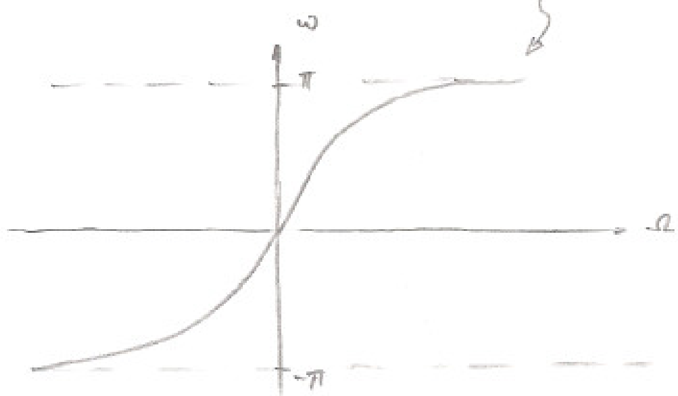
$$s\frac{T_s}{2} = j\tan\frac{\omega}{2}$$

$$(\sigma + j\Omega)\frac{T_s}{2} = j\tan\frac{\omega}{2}$$

(equate Im. parts)

$$\therefore \frac{\Omega T_s}{2} = \tan\frac{\omega}{2}$$

$$\boxed{\Omega = \frac{2\tan\frac{\omega}{2}}{T_s} \qquad \omega = 2\tan^{-1}\left(\frac{\Omega T_s}{2}\right)}$$



If $\Omega = \infty$, $\omega = \pi$
If $\Omega = 0$, $\omega = 0$.

$-\infty < \Omega < \infty$

maps within

$-\pi < \omega < \pi$

* $\boxed{\text{avoids aliasing}}$ *